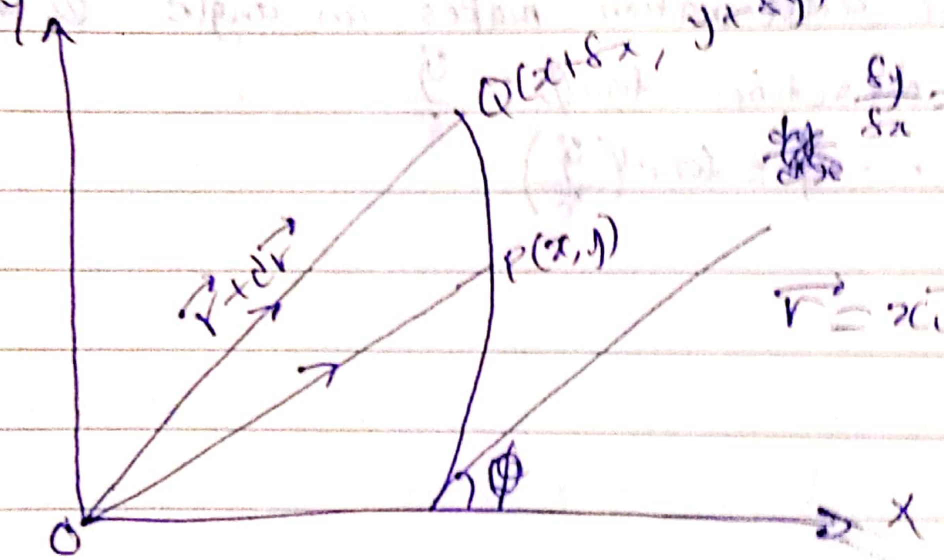
MATH 209 ANALYTICAL Dynamics 18/8/2019

- textbook of dynamics : F Chorlton
- Dynamics part II : A S Ramsey
- Dynamics of particles : S L Loney
- A Treatise on the analytical dynamics and rigid bodies : E. T. Whittaker

Mechanics < Dynamics / Statics

⇒ Dynamics deals with the action of forces on Bodies in motion.

⇒ Components of velocity and acceleration in Cartesian



$\frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{dy}{dx}$

$\vec{r} = x\vec{i} + y\vec{j}$

$$\text{velocity } (\vec{V}) = \lim_{\delta t \to 0} \frac{\delta \vec{r}}{\delta t} = \frac{\delta \vec{r}}{\delta t} = \frac{\delta(x\vec{i} + y\vec{j})}{\delta t}$$

$$= \frac{\delta x}{\delta t}\vec{i} + \frac{\delta y}{\delta t} j$$

$$\therefore \vec{V} = \dot{x}\vec{i} + \dot{y}\vec{j}$$

Magnitude of velocity (speed) $= |\vec{V}|$

$\therefore V = \sqrt{\dot{x}^2 + \dot{y}^2}$

Let ϕ be the angle which the direction of motion (velocity) make with the x axis.

for direction $\tan\phi = \frac{\delta y}{\delta x} = \frac{\frac{\delta y}{\delta t}}{\frac{\delta x}{\delta t}} = \frac{\dot{y}}{\dot{x}}$

$$\phi = \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right)$$

Acceleration $= \vec{a}\ (\text{or } \vec{f}) = \frac{\delta \vec{V}}{\delta t} = \frac{\delta}{\delta t}\left(\frac{\delta x}{\delta t}\vec{i} + \frac{\delta y}{\delta t}\vec{j}\right)$

$= \frac{\delta^2 x}{\delta t^2}\vec{i} + \frac{\delta^2 y}{\delta t^2}\vec{j}$ $\quad\therefore \vec{a} = \ddot{x}\vec{i} + \ddot{y}\vec{j}$

$a = \sqrt{\ddot{x}^2 + \ddot{y}^2}$

Let the acceleration makes an angle ϕ with the x axis then for direction $\tan\phi = \frac{\ddot{y}}{\ddot{x}}$

$$\therefore \phi = \tan^{-1}\left(\frac{\ddot{y}}{\ddot{x}}\right)$$

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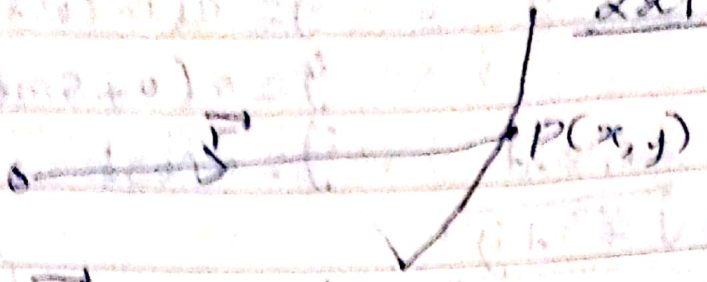
Math 509

22/8/2019



$$\vec{OP} = \vec{r} = x\vec{i} + y\vec{j}$$

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}$$

$$\boxed{V = \sqrt{\dot{x}^2 + \dot{y}^2}} \rightarrow \text{speed}$$

$$\tan\theta = \dot{y}/\dot{x}$$

$$\vec{V} = a\vec{i} + b\vec{j}$$
$$V = \sqrt{a^2 + b^2}$$

$$\vec{a} \text{ or } \vec{f} = \frac{d\vec{V}}{dt}$$

$$= \frac{d}{dt}(\dot{x}\vec{i} + \dot{y}\vec{j})$$

$$\vec{a} = \ddot{x}\vec{i} + \ddot{y}\vec{j}$$

$$\boxed{a = |\vec{a}| = \sqrt{(\ddot{x})^2 + (\ddot{y})^2}} \text{ — acceleration}$$

$$\tan\phi = \frac{\ddot{y}}{\ddot{x}} \text{ — direction}$$

Problems

Q(1) the coordinates of a moving point at time t are given by $x = a(t + \sin t)$, $y = a(1 - \cos t)$. show that it moves with constant acceleration and find the direction in which it is moving at time t.

Soln

we have

$x = a(t + \sin t)$ $\qquad$ $y = a(1 - \cos t)$

$\dot{x} = a(1 + \cos t)$ $\qquad$ $\dot{y} = a(0 + \sin t) = a \sin t$

$\ddot{x} = a(0 - \sin t) = -a \sin t$ $\qquad$ $\ddot{y} = a \cos t$

Acceleration $= f = \sqrt{\ddot{x}^2 + \ddot{y}^2}$

$= \sqrt{(-a \sin t)^2 + (a \cos t)^2}$

$= \sqrt{a^2(\sin^2 t + \cos^2 t)}$

$= \sqrt{a^2} = a$ (constant)

$\tan(90^\circ + \theta) = -\cot\theta$

Let ϕ be the angle which the direction of acceleration makes with the x-axis.

$\tan\phi = \dfrac{\ddot{y}}{\ddot{x}} = \dfrac{a \cos t}{-a \sin t} = -\cot t = \tan\left(\frac{\pi}{2} + t\right)$

$\therefore \phi = \frac{\pi}{2} + t$

Q(2) The coordinates of a moving point at time t are given by $x = a(2t + \sin 2t)$, $y = a(1 - \cos 2t)$. prove that its resultant acceleration is constant and find its direction

Soln

$\dot{x} = a(2 + 2\cos 2t) = 2a(1 + \cos 2t)$

$\ddot{x} = 2a(0 - 2\sin 2t) = -4a \sin 2t$

$\dot{y} = a(0 + 2\sin 2t)$

$\ddot{y} = a(2 \times 2 \sin 2t) = 4a \sin 2t$

$R.A = \sqrt{\ddot{x}^2 + \ddot{y}^2} = \sqrt{(4a \sin 2t)^2 + (4a \cos 2t)^2}$

$= \sqrt{16a^2(\sin^2 2t + \cos^2 2t)} = \sqrt{16a^2} = 4a$

$\tan\phi = \frac{\ddot{y}}{\ddot{x}} = \frac{4a\cos 2t}{-4a\sin 2t} = -\cot 2t = \tan\left(\frac{\pi}{2}+2t\right)$

$\therefore \phi = \frac{\pi}{2}+2t$ //

Q③ If $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ and θ increases at a uniform rate ω, prove that the velocity of the point is $a\theta\omega$ and find the inclination of the velocity to the axis of x.

Soln

$\left[\frac{d\theta}{dt} = \omega,\ \dot{\theta} = \omega\right]$

$x = a(\cos\theta + \theta\sin\theta)$

$\frac{dx}{dt} = \dot{x} = a\left[-\sin\theta\frac{d\theta}{dt} + (\theta\cos\theta + 1\cdot\sin\theta)\frac{d\theta}{dt}\right]$

$= a[-\sin\theta\,\dot{\theta} + (\theta\cos\theta + \sin\theta)\dot{\theta}]$

$= a[-\cancel{\sin\theta\,\dot{\theta}} + \dot{\theta}\theta\cos\theta + \cancel{\sin\theta\,\dot{\theta}}]$

$= a\dot{\theta}\theta\cos\theta = a\theta\omega\cos\theta$

$\frac{dy}{dt} = \dot{y} = a[\cos\theta\,\dot{\theta} - (\theta(-\sin\theta) + 1\cdot\cos\theta)\dot{\theta}]$

$= a[\cancel{\cos\theta\,\dot{\theta}} + \theta\dot{\theta}\sin\theta - \cancel{\dot{\theta}\cos\theta}]$

$= a\theta\dot{\theta}\sin\theta = a\theta\omega\sin\theta$

$V = \sqrt{\dot{x}^2 + \dot{y}^2}$

$= \sqrt{(a\theta\omega\cos\theta)^2 + (a\theta\omega\sin\theta)^2}$

$= a\theta\omega\sqrt{\cos^2\theta + \sin^2\theta}$

$\boxed{V = a\theta\omega}$ Ans.

$\tan\phi = \frac{\dot{y}}{\dot{x}} = \frac{a\theta\omega\sin\theta}{a\theta\omega\cos\theta}$

$\tan\phi = \tan\theta$

$\therefore \boxed{\phi = \theta}$ //

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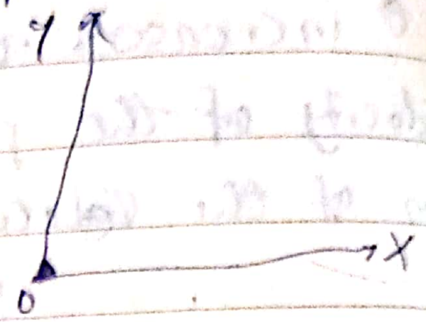
Q (i) A particle moves from a fixed point in such a way that its velocities in two fixed perpendicular directions after t seconds are $a\cos t$ and $a\sin t$ where a is constant. Prove that its acceleration be a constant magnitude and find its path.



Sln

$\dot{x} = \frac{dx}{dt} = a\cos t$ ---- (1)

$\dot{y} = \frac{dy}{dt} = a\sin t$ ---- (2)

$\therefore \ddot{x} = -a\sin t, \quad \ddot{y} = a\cos t$

Acceleration $= \sqrt{\ddot{x}^2 + \ddot{y}^2} = \sqrt{(-a\sin t)^2 + (a\cos t)^2}$

$= \sqrt{a^2(\sin^2 t + \cos^2 t)} = \sqrt{a^2} = a$

Intergrating (1) and (2)

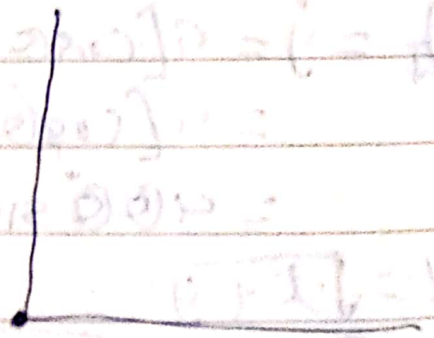
$x = a\sin t + C_1$ ---- (3) $\qquad y = -a\cos t + C_2$ ---- (4)

Using $x=0, y=0$ when $t=0$ we have

$0 = 0 + C_1, \quad 0 = -a + C_2$

$\therefore C_1 = 0, \quad C_2 = a$

Then (3) and (4) become



$x = a\sin t$

$y = -a\cos t + a$

$x = a\sin t$ and $y - a = -a\cos t$

squaring and adding we obtain

$x^2 + (y-a)^2$

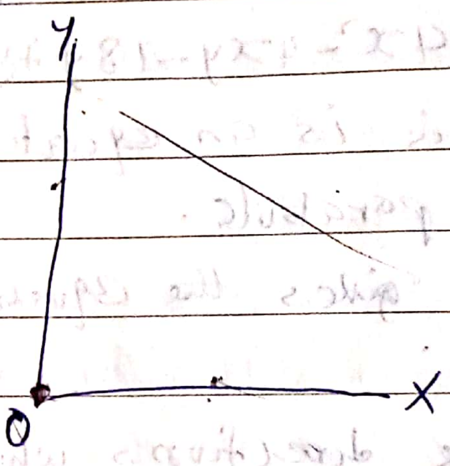
$= a^2\sin^2 t + a^2\cos^2 t$

$= a^2(\sin^2 t + \cos^2 t)$

$x^2 + (y-a)^2 = a^2$

Thus the path is a circle whose center is (a, a) and the radius is a

Q(5) A particle starts from the origin and the components of its velocity parallel to the axes of coordinate at time t are $2t+3$ and $4t$, find the path and the position of the particle when it is moving in direction equally inclined to the axes.



Sln

$\dot{x} = 2t+3$ — (1)

$\dot{y} = 4t$ — (2)

Integrating

$x = \frac{2t^2}{2} + 3t + C_1$

$x = t^2 + 3t + C_1$ — (3)

$y = \frac{4t^2}{2} = 2t^2 + C_2$

$y = 2t^2 + C_2$ — (4)

using $x=0, y=0$ when $t=0$ we have $C_1 = 0, C_2 = 0$

$\therefore x = t^2 + 3t$ — (5)

$y = 2t^2$ — (6) $\longrightarrow t^2 = y/2$

using (6) in (5) we get

$x = \frac{y}{2} + 3\sqrt{\frac{y}{2}}$

$x - \frac{y}{2} = 3\sqrt{\frac{y}{2}}$

Squaring

$(x-\frac{y}{2})^2 = \frac{9y}{2}$

$\left(\frac{2x-y}{2}\right)^2 = \frac{9y}{2}$

$\frac{(2x-y)^2}{4} = \frac{9y}{2}$

$(2x-y)^2 = 18y$

$4x^2 - 4xy + y^2 = 18y$

or $4x^2 - 4xy - 18y + y^2 = 0$

This is an equation of a parabola.

This gives the equation of the path.

The directions which are equally inclined to the x-axis are given by $\tan\theta = \pm \tan 45°$

$= \pm 1$

But $\tan\theta = \frac{\dot{y}}{\dot{x}} = \frac{4t}{2t+3}$

$\therefore \frac{4t}{2t+3} = \pm 1$

Taking +ve

$4t = 2t + 3$

$2t = 3$

$t = 3/2$

Taking −ve

putting $t = \frac{3}{2}$ in (5) & (6)

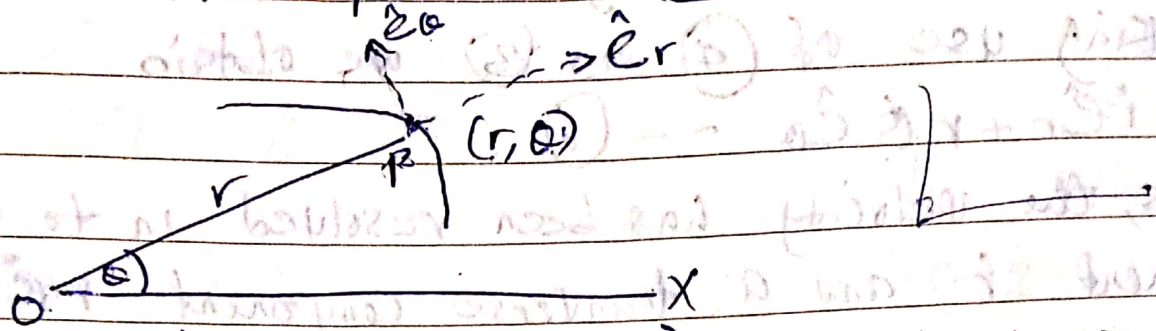
$x = \frac{9}{4} + \frac{9}{2} = \frac{9+18}{4} = \frac{27}{4}$

$y = 2 \times \frac{9}{4} = \frac{9}{2}$

$\hat{a} = \frac{\vec{a}}{a}$

$\vec{a} = a \cdot \hat{a}$

Radial and Transverse components of velocity and Acceleration ~~of a particle~~ 25/3/2019



$a \cdot b = ab\cos\phi$

$$\hat{e}_r = \cos\theta\vec{i} + \sin\theta\vec{j}$$
$$\hat{e}_\theta = -\sin\theta\vec{i} + \cos\theta\vec{j}$$
$$\hat{e}_r \cdot \hat{e}_\theta = -\cancel{\cos\theta\sin\theta} + \cancel{\sin\theta\cos\theta}$$
$$\hat{e}_r \cdot \hat{e}_\theta = 0$$

Consider a particle moving along a plane curve $\vec{r}(t)$. Let its position at any time t be at $P(r,\theta)$. Then its position vector is given by

$$\vec{r} = r\hat{e}_r \quad \text{--- (1)}$$

where $\hat{e}_r = \cos\theta\vec{i} + \sin\theta\vec{j}$ --- (2)

is a unit vector along the radial direction. The velocity of the particle is given by

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e}_r) = \frac{dr}{dt}\hat{e}_r + r\frac{d}{dt}(\hat{e}_r)$$

$$= \dot{r}\hat{e}_r + r\frac{d}{dt}(\hat{e}_r) \quad \text{--- (3)}$$

Now differentiating (2) with respect to time t, we have

$$\frac{d}{dt}(\hat{e}_r) = -\sin\theta\frac{d\theta}{dt}\vec{i} + \cos\theta\frac{d\theta}{dt}\vec{j} = (-\sin\theta\vec{i} + \cos\theta\vec{j})\dot{\theta} =$$

$$= \dot{\theta}\hat{e}_\theta \quad \text{--- (4)}$$

where $\hat{e}_\theta = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$ --- (5)
is a unit vector along the transverse direction
making use of (4) in (3) we obtain
$\vec{V} = \dot{r}\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$ --- (6)
Thus, the velocity has been resolved in to a radial component $\dot{r}$ and a transverse component $r\dot{\theta}$

The acceleration of the particle is given by
$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}[\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta] = \frac{d}{dt}(\dot{r}\hat{e}_r) + \frac{d}{dt}(r\dot{\theta}\hat{e}_\theta)$ by (6)

$= \ddot{r}\hat{e}_r + \dot{r}\frac{d}{dt}(\hat{e}_r) + \dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\frac{d\hat{e}_\theta}{dt}$

$= \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\,\hat{e}_\theta + \dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta + r\dot{\theta}\frac{d\hat{e}_\theta}{dt}$ --- (7)

Now from (5), we have $\frac{d\hat{e}_\theta}{dt} = -\cos\theta\dot{\theta}\vec{i} - \sin\theta\dot{\theta}\vec{j}$
$= -(\cos\theta\,\vec{i} + \sin\theta\,\vec{j})\,\dot{\theta}$
$= -\hat{e}_r\dot{\theta}$ --- (8) by (2)

Using (8) in (7) we have
$\vec{a} = \ddot{r}\hat{e}_r + 2\dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta + r\dot{\theta}\,(-\hat{e}_r\dot{\theta})$
$= \ddot{r}\hat{e}_r + 2\dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r$
$= (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{e}_\theta$

Thus, the acceleration has been resolved into a radial component $(\ddot{r} - r\dot{\theta}^2)$ and a transverse component $2\dot{r}\dot{\theta} + r\ddot{\theta}$

$R \cdot V = \dot{r}$

$T \cdot V = r\dot{\theta}$

$R \cdot A = \ddot{r} - r\dot{\theta}^2$

$t \cdot A = 2\dot{r}\dot{\theta} + r\ddot{\theta}$

problem

Q ① if the radial and transverse velocities of a particle are always propertional to each other, show that the equation is an equiangular spiral

Sln

$R \cdot V \propto T \cdot V$ $\qquad\qquad$ $x \propto y$

$x = ky$

$\frac{dr}{dt} \propto r\frac{d\theta}{dt}$

$\frac{dr}{dt} = \lambda r \frac{d\theta}{dt}$ $\quad$ where λ = constant

$\frac{dr}{r} = \lambda \, d\theta$

integrating

$\ln r = \lambda\theta + \ln k$

$\ln r - \ln k = \lambda\theta$

$\ln\left(\frac{r}{k}\right) = \lambda\theta$

$\frac{r}{k} = e^{\lambda\theta}$

$r = k\,e^{\lambda\theta}$ $\quad$ Thus the path is an equiangular spiral.

29/3/2019

Q(2) If the angular velocity of a point moving in a plane be constant about a fixed origin, show that ~~its~~ its transverse acceleration varies as its radial velocity.

Solution

Angular velocity $= \omega = \frac{d\theta}{dt} = \dot{\theta} =$ constant (by given condition)

$\therefore T{\cdot}A = 2\dot{r}\dot{\theta} + r\ddot{\theta} = 2\dot{r}\omega + r\cdot 0 = 2\dot{r}\omega =$ (some constant) $\dot{r}$

$\Rightarrow T{\cdot}A = K\dot{r}$ $\quad\because \ddot{\theta} = 0$ $\quad$ where $K = 2\omega$

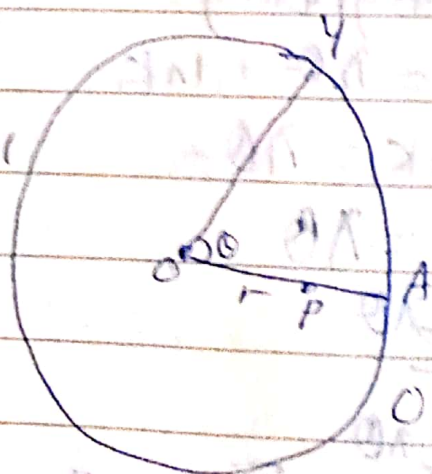
$\Rightarrow T{\cdot}A \propto \dot{r}$ (radial velocity)

Proved.

$\therefore$ T.A varies as radial velocity

Q(3) An insect crawls at a constant rate u along the spoke of a cart-wheel of radius a, the cart is moving with a constant velocity V. Find the acceleration along and perpendicular to the spoke.

Solution



$OP = r$

Let O be the centre and a be the radius of the cart-wheel. Let OA be its a spoke and the positions of the insect at time t be at P on the spoke such that $OP = r$.

Let θ be the angle which the spoke makes with the vertical direction OY.

$\therefore$ By question

$\frac{dr}{dt} = u$ (Constant) --(A)

or $dr = u\,dt$

integrating $r = ut + C$ ---(1)

Using $r=0$ when $t=0$, then $0 = 0 + C$ $\therefore C = 0$

Thus $r = ut$ — (2)

The angular velocity of the Card ~~=~~ the angular velocity of the insect $= \frac{d\theta}{dt} = \dot{\theta} = w = \frac{V}{a}$ ~~= constant~~

$V = wa$

$\therefore \dot{\theta} = \text{constant} \Rightarrow \ddot{\theta} = 0$

Also from (A), $\ddot{r} = 0$

Radial component of acceleration $= \ddot{r} - r\dot{\theta}^2 = 0 - utw^2 = -utw^2$

Acceleration along the spoke $= -utw^2$ where $w = \frac{V}{a}$

Transverse acceln

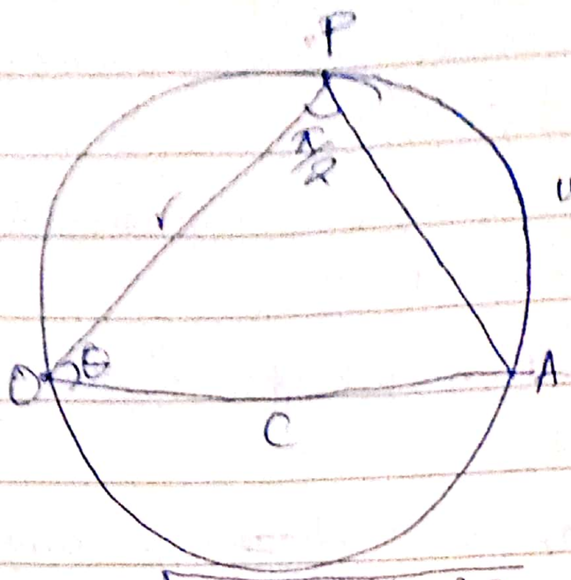
$= 2\dot{r}\dot{\theta} + r\ddot{\theta}$

$= 2uw + r.0 = 2uw$ by (A)

$\therefore$ Acceln along the perpendicular to the spoke $= 2uw$ //

Q(4) A particle moves in a circular path of radius a so that its angular velocity about a fixed point in the circumference of the circle is constant and equal to w. Show that the resultant acceleration of the particle at every point of the path is of constant magnitude $4aw^2$

Solution



O = Centre

OC = AC = a

$\omega = \frac{d\theta}{dt} = \dot{\theta} = \text{constant}$

$\therefore \ddot{\theta} = 0$

Resultant Accel^n $= \sqrt{(R\cdot A)^2 + (T\cdot A)^2}$

we take O as a fixed point in the ~~Circular path~~ circumferance of the circular path.

OA is the diameter Let P be the position of the particle at any time t. Let $\angle AOP = \theta$

In ΔAOP, $\cos\theta = \frac{OP}{OA} = \frac{r}{2a}$

$\therefore r = 2a\cos\theta$

$\dot{r} = -2a\sin\theta\,\dot{\theta} = -2a\sin\theta\,\omega = -2a\omega\sin\theta$

$\ddot{r} = -2a\omega\cos\theta\,\dot{\theta} = -2a\omega^2\cos\theta$

$R\cdot A = \ddot{r} - r\dot{\theta}^2$

$= -2a\omega^2\cos\theta - 2a\cos\theta\,\omega^2$

$= -4a\omega^2\cos\theta$

$T\cdot A = 2\dot{r}\dot{\theta} + r\ddot{\theta}$

$= 2(-2a\omega\sin\theta)\omega + r\cdot 0$

$= -4a\omega^2\sin\theta$

$\therefore$ Resultant acceleration $= \sqrt{(R\cdot A)^2 + (T\cdot A)^2}$

$= \sqrt{(-4a\omega^2\cos\theta)^2 + (-4a\omega^2\sin\theta)^2}$

$$= \sqrt{16a^2w^4(\cos^2\theta + \sin^2\theta)}$$
$$= \sqrt{16a^2w^4}$$
$$= 4aw^2$$ Proved

Q(5) A particle describes an equiangular spiral $r = ae^{m\theta}$ with a constant velocity. Find the components of the velocity and the acceleration along the radius vector and perpendicular to it.

Soln

$\because$ $r = ae^{m\theta}$ --- (1)

$\dot{r} = am\,e^{m\theta}\cdot\dot{\theta} = rm\dot{\theta}$ ~ (2)

Let the constant velocity of the particle $= V$

$\therefore V^2 = (R.V)^2 + (T.V)^2$

$= \dot{r}^2 + (r\dot{\theta})^2$

$= r^2m^2\dot{\theta}^2 + r^2\dot{\theta}^2$

$V^2 = r^2\dot{\theta}^2(m^2+1)$

$\therefore r^2\dot{\theta}^2 = \dfrac{V^2}{(m^2+1)}$

or $r\dot{\theta} = \dfrac{V}{\sqrt{1+m^2}}$ = constant --- (3)

(i) $R.V = \dot{r} = rm\dot{\theta}$ [by (2)]

$= \dfrac{mV}{\sqrt{1+m^2}}$ [using (3)]

(ii) $T.V = r\dot{\theta} = \dfrac{V}{\sqrt{1+m^2}}$ (by (3))

(iii) $R\cdot A = \ddot{r} - r\dot{\theta}^2 = 0 - \frac{(r\dot{\theta})^2}{r} = -\frac{1}{r}\left(\frac{V^2}{1+m^2}\right)$ [by (3)]

(iv) $T\cdot A = 2\dot{r}\dot{\theta} + r\ddot{\theta}$

$$= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

$$= \frac{1}{r}\frac{d}{dt}(r\cdot r\dot{\theta}) = \frac{1}{r}\frac{d}{dt}\left[r\cdot\frac{V}{\sqrt{1+m^2}}\right] \text{ [by (3)]}$$

$$= \frac{V}{r\sqrt{1+m^2}}\frac{dr}{dt} = \frac{V}{r\sqrt{1+m^2}}\dot{r} = \frac{V}{r\sqrt{1+m^2}}\cdot\frac{mV}{\sqrt{1+m^2}} \text{ [by (1)]}$$

$$= \frac{mV^2}{r(1+m^2)} \quad //$$

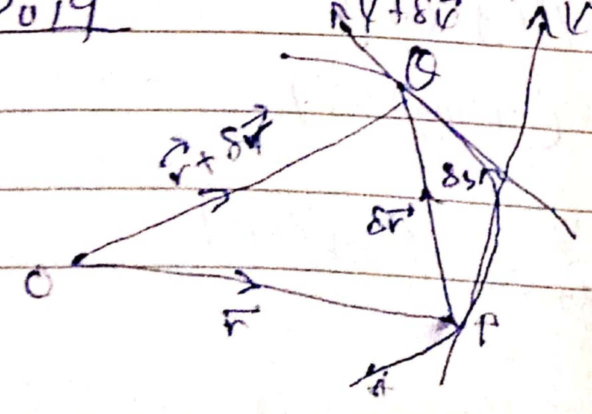
Q(6) A point describes with a constant velocity obtain O, the spiral $r = e^{\theta}$, O being the pole of the spiral. find the radial and component acceleration

Sln

Ans $R\cdot A = -\frac{V^2}{2r}$, $T\cdot A = \frac{V^2}{2r}$

Hint put in Q(5) $a=1$, $m=1$

Tangential and Normal components of Velocity and Acceleration 1/4/2019



arc $AP = s$

arc $AQ = s+\delta s$

Consider a particle moving along a plane curve. Let its position at time t and $t+\delta t$ be P and Q such that $\vec{OP} = \vec{r}$, $\vec{OQ} = \vec{r} + \delta\vec{r}$, where O is the origin. Let A be a fixed point on the curve and arc $AP = s$, arc $AQ = s+\delta s$.

Then the velocity of the particle at the point P is

$$\vec{V} = \lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \dot{s}\frac{d\vec{r}}{ds} \quad (1) \quad \text{with } \dot{s} = \frac{ds}{dt}.$$

Now $\frac{d\vec{r}}{ds} = \lim_{\delta s \to 0} \frac{\delta\vec{r}}{\delta s}$

Since $\frac{\delta\vec{r}}{\delta s}$ is a vector from P to Q and when $\delta s \to 0$ then $\delta\vec{r} = \vec{PQ}$ tends to that tangent at P to the curve.

Let $\hat{T}$ be a unit vector in the direction of this tangent.

Then $$\frac{d\vec{r}}{ds} = \left|\lim_{\delta s \to 0} \frac{\delta\vec{r}}{\delta s}\right|\hat{T} = \left|\lim_{Q \to P} \frac{\text{chord } PQ}{\text{arc } PQ}\right|\hat{T} = 1\cdot\hat{T} = \hat{T} \quad (2)$$

making use of (2), (1) reduces to

$$\vec{V} = \dot{s}\hat{T} \quad (3)$$

Thus the velocity of the particle is along the direction of tangent at P and has only tangential component $\dot{s}$.

The acceleration of the particle is

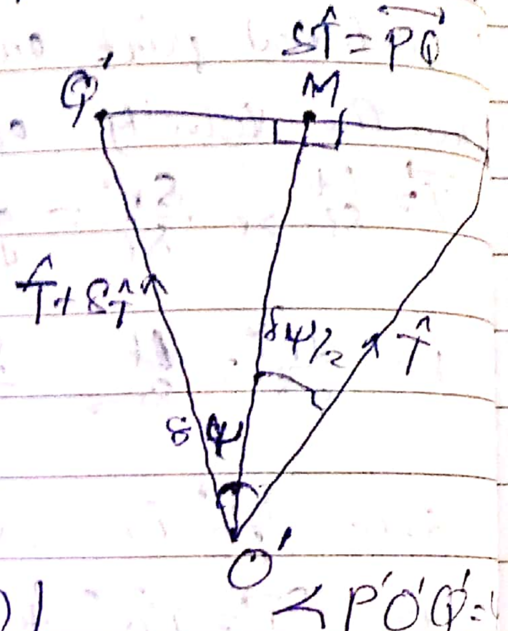
$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}(\dot{s}\hat{T}) = \frac{d}{dt}(\dot{s})\hat{T} + \dot{s}\frac{d\hat{T}}{dt} = \ddot{s}\hat{T} + \dot{s}\frac{d\hat{T}}{ds}\frac{ds}{dt}$$

$$= \ddot{s}\hat{T} + \dot{s}^2\frac{d\hat{T}}{ds} \quad (4)$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} \Rightarrow \vec{a} = \hat{a}|\vec{a}|$$

To evaluate $\frac{d\hat{T}}{ds}$ we interpret $\hat{T}+\delta\hat{T}$ to mean a unit tangent vector at Q.

We consider triangle $O'P'Q'$. $O'P'$ and $O'Q'$ have unit length.



$P'Q' = |\delta\hat{T}|$. M is the middle point of $P'Q'$ and $\angle P'O'Q' = \psi$

$$\therefore \sin\frac{\psi}{2} = \frac{P'M}{O'P'} = \frac{P'M}{1}$$

$$\Rightarrow P'M = \sin\psi/2$$

$$\therefore P'Q' = 2P'M = 2\sin\psi/2 = |\delta\hat{T}|$$

Now $\left|\frac{d\hat{T}}{ds}\right| = \left|\lim_{\delta s\to 0}\frac{\delta\hat{T}}{\delta s}\right| = \lim_{\delta s\to 0}\left|\frac{2\sin(\delta\psi/2)}{\delta s}\right|$ $\quad \lim_{\delta s\to 0}\frac{P'Q'}{\delta s}$

$$= \lim_{\delta s\to 0} 2\left(\frac{\sin\delta\psi/2}{\psi/2}\right)\cdot 2\left(\frac{\delta\psi}{\delta s}\right) = 1\cdot\frac{d\psi}{ds} = \frac{1}{\rho}$$

where $\rho = \frac{ds}{d\psi}$ -- (5)

is the radius of the curvature.

since $|\hat{T}|=1$ and $\frac{d\hat{T}}{ds}$ is perpendicular to $\hat{T}$, therefore let $\hat{N}$ be a unit normal vector perpendicular to $\hat{T}$ and directed toward the centre of the curve.

Hence $\frac{d\hat{T}}{ds} = \left|\frac{d\hat{T}}{ds}\right|\hat{N} = \frac{\hat{N}}{\rho}$ -- (6) (by (5))

from (4) and (6) we have $\vec{a} = \ddot{s}\hat{T} + \frac{\dot{s}^2}{\rho}\hat{N}$ -- (7)

Hence the acceleration has a tangential components $\ddot{s}$ and a normal components $\frac{\dot{s}^2}{\rho}$ //

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$T \cdot V = \dot{s}$ $\quad N \cdot A = \frac{\dot{s}^2}{\rho}$, $\quad \rho = \frac{ds}{d\psi}$

$T \cdot A = \ddot{s}$ $\quad \dot{s} = \frac{ds}{dt}$

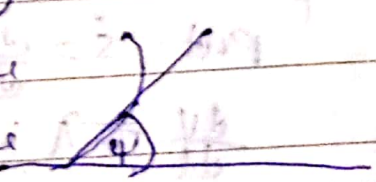
Problems

Q(1) A point moves in a plane curve so that its tangential and normal accelerations are equal and the tangent rotates with constant velocity find the path.

Sln

Given that $T \cdot A = N \cdot A$ $\quad \frac{d\psi}{dt} = k$, say = constant

where ψ is the angle which the tangent makes with the fixed line.



i.e $\ddot{s} = \frac{\dot{s}^2}{\rho}$

let $\dot{s} = \frac{ds}{dt} = v$

$\frac{dv}{dt} = \frac{v^2}{\rho}$

$\frac{dv}{ds}\frac{ds}{dt} = v^2 \frac{d\psi}{ds}$

$v\frac{dv}{ds} = v^2 \frac{d\psi}{ds}$

$\frac{dv}{v} = d\psi$

Integrating

$\ln v = \psi + \ln A$

$\ln v - \ln A = \psi$

$\ln\left(\frac{v}{A}\right) = \psi$

$\frac{v}{A} = e^{\psi}$

$v = Ae^{\psi}$

$\frac{ds}{dt} = Ae^{\psi}$

$\frac{ds}{d\psi}\frac{d\psi}{dt} = Ae^{\psi}$

$k\frac{ds}{d\psi} = Ae^{\psi}$

$k\,ds = Ae^{\psi}d\psi$

Integrating

$ks = Ae^{\psi} + B$

This is the equation of the path.

f: $\frac{dv}{dt} = v\frac{dv}{ds} = \frac{d^2s}{dt^2}$

Q(2) If the tangential and the normal acceleration of a particle describing a plane curve be constant throughout its motion, prove that the radius of curvature at any time t is given by $\rho = (At+B)^2$, where A and B are constant.

Sln

$\ddot{s} = \lambda$ say (constant) (1)

$\frac{\dot{s}^2}{\rho} = \mu$ say (constant) (2)

put $\dot{s} = \frac{ds}{dt} = v$ then

$\frac{dv}{dt} = \lambda$ -- (3) and $\frac{v^2}{\rho} = \mu$ ~ (4)

from (4) $\rho = \frac{v^2}{\mu} = \left(\frac{v}{\sqrt{\mu}}\right)^2$ -- (5)

from (3) $dv = \lambda\, dt$

integrating $v = \lambda t + k$ - (6)

from (5) and (6)

$\rho = \left(\frac{\lambda t + k}{\sqrt{\mu}}\right)^2 = \left(\frac{\lambda}{\sqrt{\mu}}t + \frac{k}{\sqrt{\mu}}\right)^2 = (At+B)^2 \Rightarrow$ proved

where $\frac{\lambda}{\sqrt{\mu}} = A$, $\frac{k}{\sqrt{\mu}} = B$.

Q(3) A point describes the curve $s = 4a\sin\psi$ with uniform speed v. find the acceleration at any point

Sln

$s = 4a\sin\psi$ -- (1) $\quad \dot{s} = v$ (uniform) -- (2)

$\therefore \frac{ds}{d\psi} = 4\cos\psi$ $\quad \ddot{s} = 0$

or $\rho = 4\cos\psi$ -- (3)

Resultant acceleration

$$f = \sqrt{(T\cdot A)^2 + (N\cdot A)^2}$$

$$= \sqrt{(\ddot{S})^2 + \left(\frac{\dot{S}^2}{\rho}\right)^2}$$

$$= \sqrt{0 + \left(\frac{\dot{S}^2}{\rho}\right)^2}$$

$$= \frac{\dot{S}^2}{\rho} = \frac{V^2}{4a\cos\psi}$$ toward the normal by (2) & (4)

Q(4) prove that the angular acceleration of the direction of motion of a point moving in a plane is

$$\frac{V}{\rho}\frac{dV}{ds} - \frac{V^2}{\rho^2}\frac{d\rho}{ds}$$

Sln

The angular acceleration of the direction of motion is

$$\frac{d^2\psi}{dt^2} = \frac{d}{dt}\left(\frac{d\psi}{dt}\right) = \frac{d}{dt}\left(\frac{d\psi}{ds}\cdot\frac{ds}{dt}\right) = \frac{d}{dt}\left(V\frac{d\psi}{ds}\right)$$

$$= \frac{d}{dt}\left(\frac{V}{\rho}\right) = \frac{1}{\rho}\frac{dV}{dt} - \frac{1}{\rho^2}\frac{d\rho}{dt}V = \frac{1}{\rho}\frac{dV}{ds}\frac{ds}{dt} - \frac{1}{\rho^2}\frac{d\rho}{ds}\frac{ds}{dt}V$$

$$= \frac{V}{\rho}\frac{dV}{ds} - \frac{V^2}{\rho^2}\frac{d\rho}{ds}$$ proved $\left[\because \frac{ds}{dt} = V\right]$

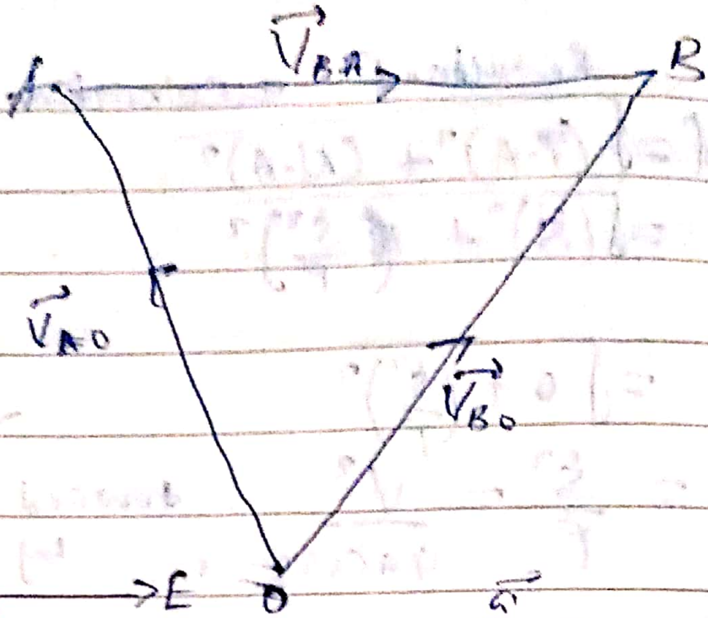
Relative Velocity

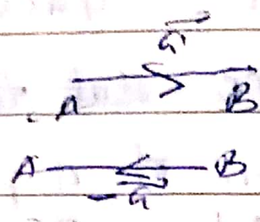
The velocity of a point is relative to that of a point A is the rate of change of displacement of is w.r.t A. The relative velocity of B w.r.t A is denoted by $\vec{V}_{BA}$

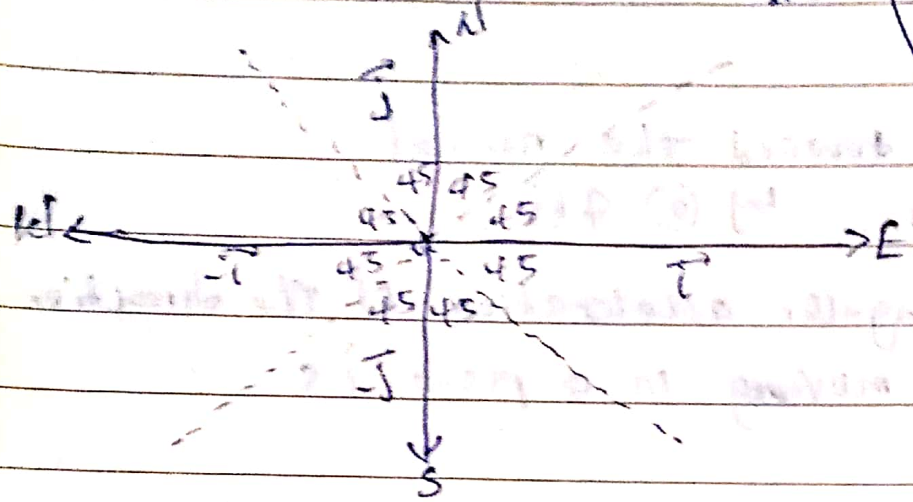
$\vec{OA} + \vec{AB} = \vec{OB}$

$\vec{AB} = \vec{OB} - \vec{OA}$

$$\boxed{\vec{V}_{BA} = \vec{V}_{BO} - \vec{V}_{AO}}$$

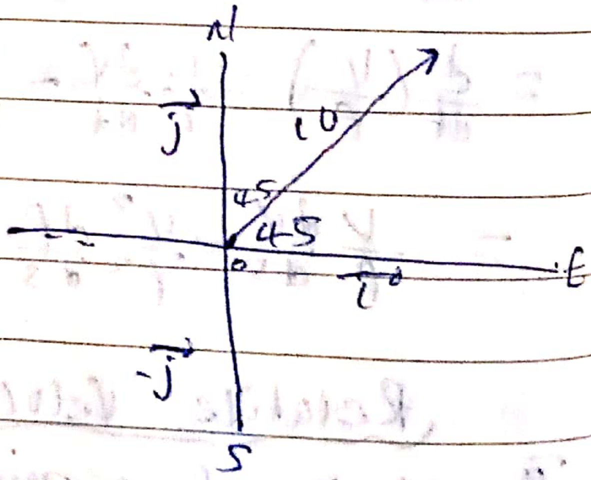






Problems

Q(1) A ship is sailing north-east with a velocity of 10km/hour and to a passenger on board the ~~wt~~ wind appears to blow from north with a velocity of $10\sqrt{2}$ km/hr. Find the true velocity of the wind.



Soln

Let $\hat{i}$ and $\hat{j}$ denote the velocity ~~be unit vector~~ of 1km/hr toward east and north direction respectively.

Let $\vec{V}_{sh}$ = velocity of the ship

$\vec{V}_w$ = True velocity of the wind

$\therefore V_{sh} = 10\cos 45^\circ \vec{i} + 10\sin 45^\circ \vec{j}$

$\therefore V_{sh} = \frac{10}{\sqrt{2}}\vec{i} + \frac{10}{\sqrt{2}}\vec{j}$

$V_w - V_{sh} = -10\sqrt{2}\, j$

$\therefore V_w = V_{sh} - 10\sqrt{2}\, j$

$= \frac{10}{\sqrt{2}} i + \frac{10}{\sqrt{2}} j - 10\sqrt{2}\, j$

$= \frac{10}{\sqrt{2}} i + \left(\frac{10}{\sqrt{2}} - 10\sqrt{2}\right) j$

$= \frac{10}{\sqrt{2}} \vec{i} + \left(\frac{10-20}{\sqrt{2}}\right) \vec{j}$

$\vec{V_w} = \frac{10}{\sqrt{2}} \vec{i} - \frac{10}{\sqrt{2}} \vec{j}$

$V_w = \sqrt{\left(\frac{10}{\sqrt{2}}\right)^2 + \left(\frac{-10}{\sqrt{2}}\right)^2}$

$= \sqrt{50+50} = 10$ km/h

for direction:

Toward south-east

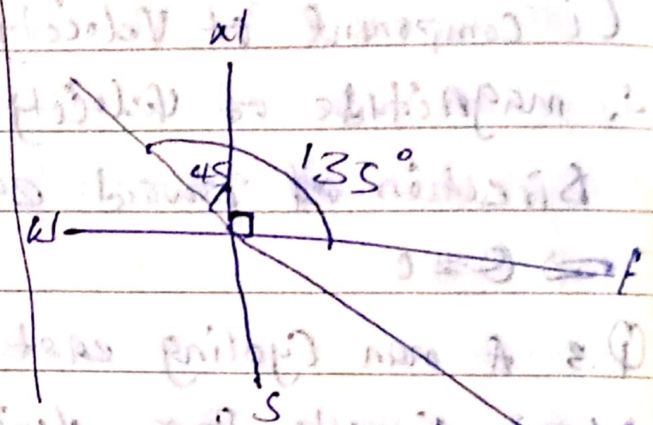
or from N-W

OR

$\tan\theta = \frac{-10\sqrt{2}}{10\sqrt{2}} = -1$

$= \tan 135°$

$\theta = 135°$



8/4/2019

Q2 A man in a ship steaning north-east at 10km/hr observes the smoke is ~~…~~ using from the funnel in a south-east direction. He estimates the speed of the smoke to be the same as that of the ship. what is the magnitude and direction of the velocity of the wind?

Sln

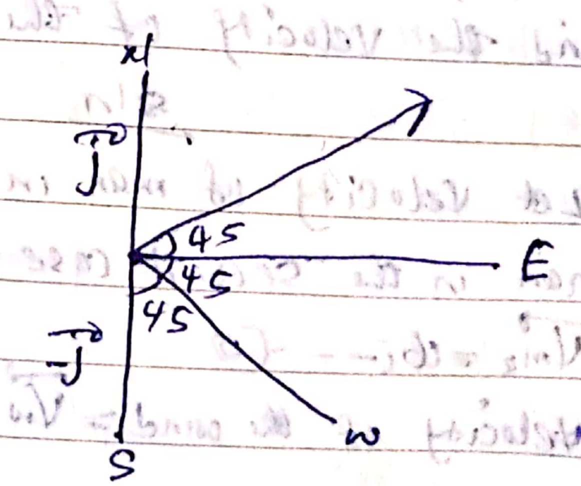
Let the velocity of the ship $= \vec{V_{sh}}$

Velocity of the smoke (wind) $= \vec{V_w}$

Then we have

$\vec{V_{sh}} = 10\cos 45° \vec{i} + 10\sin 45 \vec{j}$

$= \frac{10}{\sqrt{2}} \vec{i} + \frac{10}{\sqrt{2}} \vec{j}$



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$\vec{V_w} - \vec{V_{sh}} = \frac{10\vec{i}}{\sqrt{2}} - \frac{10}{\sqrt{2}}j$

$\therefore \vec{V_w} = \vec{V_{sh}} + \frac{10}{\sqrt{2}}i - \frac{10}{\sqrt{2}}j$

$= \frac{10\vec{i}}{\sqrt{2}} + \frac{10}{\sqrt{2}}j + \frac{10\vec{i}}{\sqrt{2}} - \frac{10}{\sqrt{2}}j$

$= \frac{10}{\sqrt{2}}i + \frac{10\vec{i}}{\sqrt{2}} = \frac{20\vec{i}}{\sqrt{2}} = 10\sqrt{2}\vec{i}$

($\vec{i}$ component of velocity is toward east).

$\therefore$ magnitude of velocity $= 10\sqrt{2}$ km/h

Direction ~~of~~ Toward east $\tan\theta = \frac{0}{\frac{10}{\sqrt{2}}}$ $\therefore \tan\theta = \tan 0$

$\Rightarrow \theta = 0$

Q3 A man cycling east at 8km/h find the wind appears to blow directly from north on doubling his speed it appear to blow from the N-E. find the actual velocity of the wind.

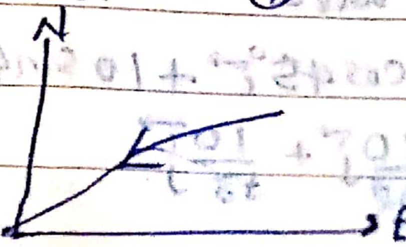
Q4. A person travelling due east of 4km/h find the wind appear to blow directly from the north. when he double his speed the wind appear to come from the north-east find the velocity of the wind [Ans: $4\sqrt{2}$ km/h from N-W]

sln

Let velocity of man in the first case $= \vec{V_{m_1}}$, let velocity of man in the second case $= \vec{V_{m_2}}$ $\vec{V_{m_1}} = 8\vec{i}$ --- (1)

$\vec{V_{m_2}} = 16i$ --- (2)

velocity of the wind $= \vec{V_w}$



let $\vec{V_w} + \vec{V_{m_1}} = -\lambda\vec{j}$ — (3)

$V_w - V_{m_2} = -u\cos 45^\circ i - u\sin 45^\circ j$ — (4)

from N-E it is $-\vec{i}$

$\vec{V_w} - \vec{V_{m_2}} = -\frac{u}{\sqrt{2}}\vec{i} - \frac{u}{\sqrt{2}}\vec{j}$

subtracting (4) from (3)

$\vec{V_{m_2}} - \vec{V_{m_1}} = \frac{u}{\sqrt{2}}\vec{i} + \frac{u}{\sqrt{2}}\vec{j} - \lambda\vec{j}$

$\vec{V_{m_2}} - \vec{V_{m_1}} = \frac{u}{\sqrt{2}}i + \left(\frac{u}{\sqrt{2}} - \lambda\right)j$

$16\vec{i} - 8\vec{i} = \frac{u}{\sqrt{2}}i$

equating the coefficient of $\vec{i}$ and j on both side we have $\frac{u}{\sqrt{2}} = 8i$ $\quad \frac{u}{\sqrt{2}} - \lambda = 0$

$\therefore u = 8\sqrt{2} \quad \lambda = 8$

making use of (5) in (3)

$\vec{V_w} = \vec{V_{m_1}} - \lambda\vec{j} = 8\vec{i} - 8\vec{j}$

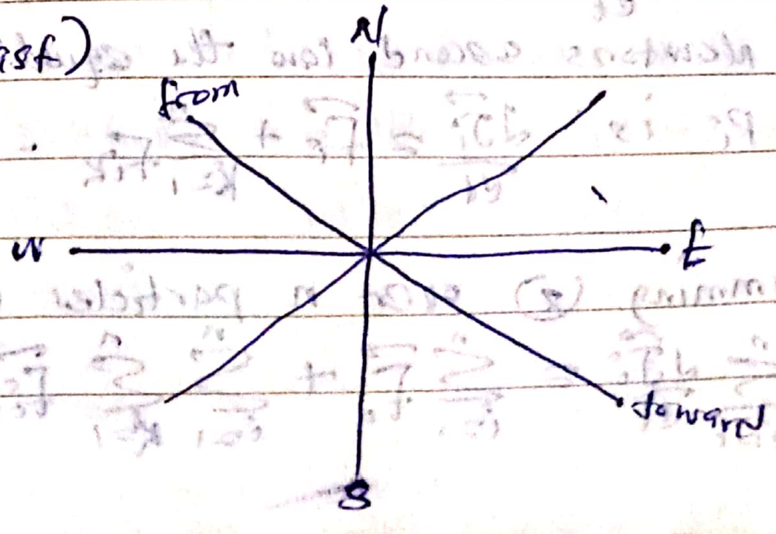
magnitude of velocity

$= \sqrt{8^2 + (-8)^2} = \sqrt{64+64} = \sqrt{2 \times 64} = 8\sqrt{2}$ Km/h

for Direction from N-W or toward S-E

$\tan\theta = \frac{8}{-8} = \tan^{-1}(-1) \Rightarrow \theta = \tan^{-1}(-1)$

$\theta = -45^\circ$ (from North east)



Conservation Theorem for linear momentum 12|4|2019

Statement: If the sum of external forces acting on a system of particles be ~~be~~ zero in any direction then the total linear moment of the system in that direction remains constant throughout the motion.

Proof

Consider a system of n particle P_i $(i = 1, 2 - - n)$ each with a constant mass m_i and position vector $\vec{r_i}$ at any time t with respect to an origin of the inertial frame. The forces as applied to the particle P_i consist of an external forces $\vec{F_i}$ and the external forces due to interaction among particles.

The internal force $\vec{F_{ik}}$ which the particle P_k exerts on the particle P_i is equal and opposite to the internal force $\vec{F_{ki}}$ which the particle P_i exerts on the particle P_k.

These internal forces are collinear, therefore we have

$$\vec{F_{ik}} + \vec{F_{ki}} = 0 \quad --(1)$$

The linear momentum of the particle P_i is given by

$$\vec{J_i} = m_i \frac{d\vec{r_i}}{dt}$$

By Newtons second law the equation of motion of the particle P_i is

$$\frac{d\vec{J_i}}{dt} = \vec{F_i} + \sum_{k=1}^{n} \vec{F_{ik}} \quad --(2)$$

summing (2) over n particles of the system, we have

$$\sum_{i=1}^{n} \frac{d\vec{J_i}}{dt} = \sum_{i=1}^{n} \vec{F_i} + \sum_{i=1}^{n}\sum_{k=1}^{n} \vec{F_{ik}} \quad --(3)$$

By (1) $\sum_{i=1}^{n}\sum_{k=1}^{n}\vec{f_{ik}} = 0$

Now, let $\sum_{i=1}^{n} \vec{f_i} = \vec{F}$ = total force acting on the system

$\sum_{i=1}^{n} \vec{J_i} = \vec{J}$ = Total linear momentum of the system

Then, from (3) we have $\frac{d\vec{J}}{dt} = \vec{F}$

~~a~~ If no external forces act on the system

i.e $\vec{F} = 0$, then $\frac{d\vec{J}}{dt} = 0 \Rightarrow d\vec{J} = 0$

Integrating $\vec{J}$ = constant

Hence the theorem.

$\square$

Conservation Theorem for Angular Momentum:

Statement: If the sum of external torques about a fixed point acting on a system of particles be zero then the total angular momentum of the system about that fixed point remains constant throughout the motion.

proof

Consider a system of n particles P_i ($i = 1, 2 --- n$) in motion each with a constant mass m_i and position vector $\vec{r_i}$ at any time t. Then the equation of motion of the i-th particle P_i is given by

$$m_i \frac{d^2\vec{r_i}}{dt^2} = \vec{f_i} + \sum_{k=1}^{n} \vec{f_{ik}} \quad --(1)$$

r^{th}

where $\vec{F_i}$ is the external force acting on the system and $\vec{f_{ik}}$ is the internal force due to interaction among particles.

Taking the vector product of ① with $\vec{r_i}$ we have

$$\vec{r_i} \times m_i \frac{d^2\vec{r_i}}{dt^2} = \vec{r_i} \times \vec{F_i} + \sum_{k=1}^{n} \vec{r_i} \times \vec{f_{ik}} \quad \text{--- ②}$$

The angular momentum of P_i is given by

$$\vec{H_i} = \vec{r_i} \times m_i \frac{d\vec{r_i}}{dt}$$

$$\therefore \frac{d\vec{H_i}}{dt} = \frac{d\vec{r_i}}{dt} \times m_i \frac{d\vec{r_i}}{dt} + \vec{r_i} \times m_i \frac{d^2\vec{r_i}}{dt^2} = 0 + \vec{r_i} \times m_i \frac{d^2\vec{r_i}}{dt^2}$$

$[\because \vec{a} \times \vec{a} = 0]$

$$\therefore \frac{d\vec{H_i}}{dt} = \vec{r_i} \times m_i \frac{d^2\vec{r_i}}{dt^2} \quad \text{--- ③}$$

from ② and ③ we have

$$\frac{d\vec{H_i}}{dt} = \vec{r_i} \times \vec{F_i} + \sum_{k=1}^{n} \vec{r_i} \times \vec{f_{ik}}$$

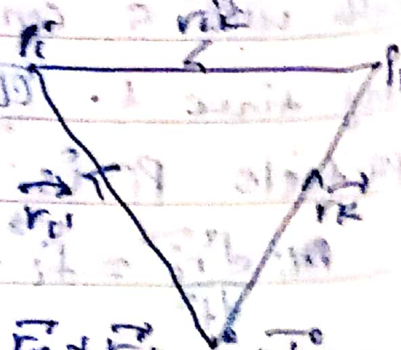
summing it over n particles we ~~get~~ get

$$\sum_{i=1}^{n} \frac{d\vec{H_i}}{dt} = \frac{d\vec{H}}{dt} = \sum_{i=1}^{n} \vec{r_i} \times \vec{F_i} + \sum_{i=1}^{n} \sum_{k=1}^{n} f_{ik} \quad \text{--- ④}$$

Since the internal forces $\vec{f_{ik}}$ act along the line joining the particles, therefore the sum of internal torques is zero. To see this we have

$$\vec{r_i} \times \vec{f_{ik}} + r_k \times f_{ki} = \vec{r_i} \times \vec{f_{ik}} - r_k \times \vec{f_{ik}}$$

$$= (\vec{r_i} - \vec{r_k}) \times \vec{f_{ik}} = \vec{r_{ik}} \times \vec{f_{ik}} = 0$$



$\therefore r_{ik}$ is parallel to the line of action of $\vec{f_{ik}}$

$\because \vec{r_k} + \vec{r_{ik}} = \vec{r_i}$

$\therefore \vec{r_{ik}} = \vec{r_i} - \vec{r_k}$

Hence $\displaystyle\sum_{i=1}^{n} \sum_{k=1}^{n} \vec{f_{ik}} = 0$

Then (4) becomes

$$\frac{d\vec{H}}{dt} = \sum_{i=1}^{n} \vec{r_i} \times \vec{f_i}$$

If the sum of external torques is zero i.e $\sum_{i=1}^{n} \vec{r_i} \times \vec{f_i} = 0$

then we have

$$\frac{d\vec{H}}{dt} = 0 \Rightarrow d\vec{H} = 0$$

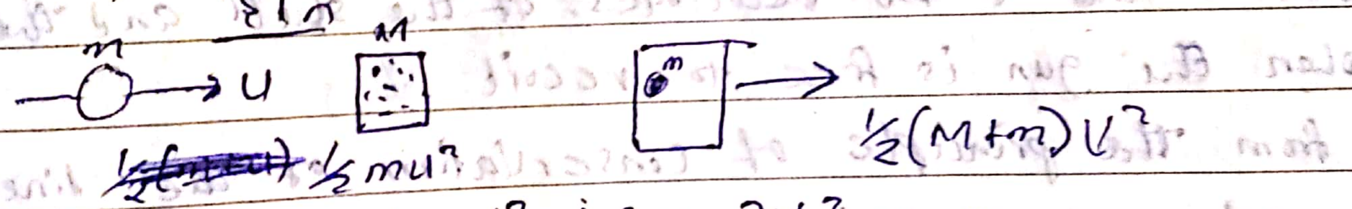
Integrating $\vec{H}$ = constant

Hence the theorem.

Problems 15/4/2019

(1) A bullet of mass m, moving with velocity u, strikes a block of mass M, which is free to move in the direction of bullet, and is embedded in it. show that the loss in the kinetic energy is $\frac{1}{2}\left(\frac{Mmu^2}{M+m}\right)$

Sln



Loss in K.E $= \frac{1}{2}mu^2 - \frac{1}{2}(M+m)V^2$ --- (1)

where V is the common velocity of the block when the bullet is embedded in it.

from the principle of conservation of the linear momentum.

$$mu + M \cdot 0 = (M+m)V$$

$$\therefore V = \frac{mu}{M+m} \quad \text{--- (2)}$$

making use of (2) in (1), then loss in

$$K.E = \frac{1}{2}mu^2 - \frac{1}{2}(M+m)\frac{m^2u^2}{(M+m)^2}$$

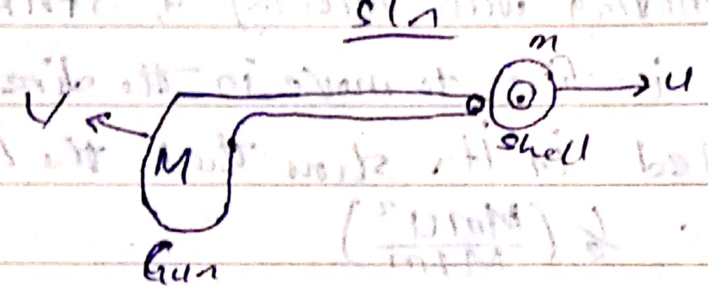
$= \frac{1}{2} m u^2 \left[1 - \frac{m}{M+m}\right]$

$= \frac{1}{2} m u^2 \left[\frac{M+m-m}{M+m}\right]$

$= \frac{1}{2} \frac{m M u^2}{(M+m)}$ Proved.

Q(2) A shell of mass m is ejected from a given of mass M by an explosion which generates kinetic energy E. prove that the initial velocity of the shell is

$\sqrt{\frac{2ME}{m(m+M)}}$



Let u and V be velocities of the shell and the gun when the gun is free to recoil

from the principle of conservation of the linear momentum $mu - MV = 0$

or $MV = mu$

$V = \frac{mu}{M}$ --- (1)

Also $E = \frac{1}{2} m u^2 + \frac{1}{2} M V^2$

$= \frac{1}{2} m u^2 + \frac{1}{2} M \frac{m^2 u^2}{M^2}$ by [(1)]

$= \frac{1}{2} m u^2 + \frac{1}{2} \frac{m^2 u^2}{M}$

$E = \frac{1}{2} m u^2 \left[1 + \frac{m}{M}\right]$

$$E = \frac{1}{2} mu^2 \left(\frac{M+m}{M} \right)$$

or $2ME = mu^2(M+m)$

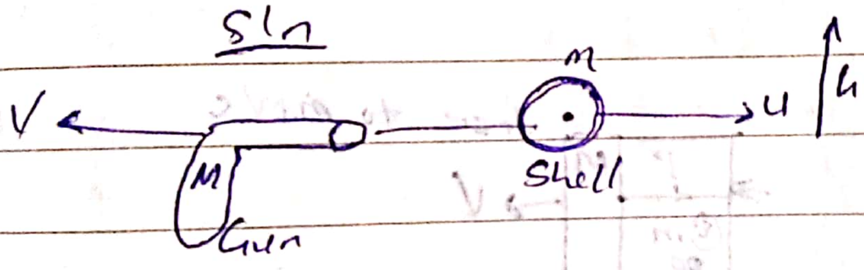
or $u^2 = \frac{2ME}{m(M+m)}$

$$u = \sqrt{\frac{2ME}{m(M+m)}}$$

proved

Q③ A gun of mass M, fires a shell of mass m horizontally and the energy of explosion is such as would be sufficient to projected the shell vertically to a height h. Show that the velocity of recoil of the gun is $\sqrt{\frac{2m^2gh}{M(M+m)}}$

Sln



Let u and V be velocities of the shell and the gun when the gun is free to recoil.

from the principal of Conservation of linear momentum

$MV = mu \quad \therefore u = \frac{MV}{m}$ --- ①

Let E be energy generated by explosion

$\therefore E = \frac{1}{2} mu^2 + \frac{1}{2} MV^2$

$= \frac{1}{2} m \left(\frac{M^2V^2}{m^2} \right) + \frac{1}{2} MV^2$ (by ①)

$= \frac{1}{2} MV^2 \left(\frac{M}{m} + 1 \right)$

$\therefore E = \frac{1}{2}MV^2\left(\frac{M+m}{m}\right)$ --- (3)

from (2) and (3) we have

$\frac{1}{2}MV^2\left(\frac{M+m}{m}\right) = mgh$

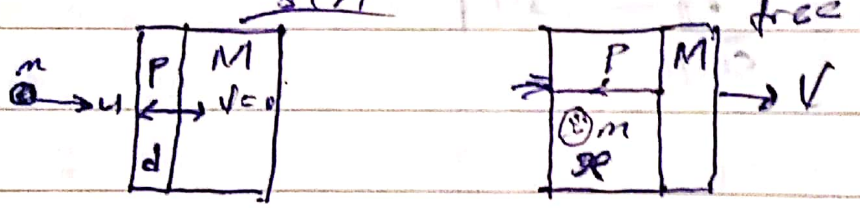
or $M(M+m)V^2 = 2m^2gh$

or $V^2 = \frac{2m^2gh}{M(M+m)}$

or $V = \sqrt{\frac{2m^2gh}{M(M+m)}}$ proved.

Q(4): A shot of mass m penetrates a thickness d of a fixed plank of mass M. If M were free to move, show that the thickness penetrated would be $\frac{Md}{M+m}$ assuming the resistance of the plank to be uniform.

Sln



Let u be the velocity of shot just before entering the plank

Let P be the uniform resistance of the plank

$\because$ Loss in kinetic energy = work done

$\therefore \frac{1}{2}mu^2 - 0 = Pd$ --- (1)

In the second case

Loss in K. E = work done

$\frac{1}{2}mu^2 - \frac{1}{2}(M+m)V^2 = Px$ --- (2)

where V is the common velocity of the plank and the shot after penetrating a distance x. Also from the principle of conservation of linear momentum

$$mu = (M+m)V$$

$$\therefore V = \frac{mu}{M+m} \quad \text{---} \; (3)$$

using (3) in (2) we have

$$\frac{1}{2}mu^2 - \frac{1}{2}(M+m)\frac{m^2u^2}{(M+m)^2} = Px$$

$$\frac{1}{2}mu^2\left(1 - \frac{m}{M+m}\right) = Px$$

$$\frac{1}{2}mu^2\left(\frac{M+\cancel{m}-\cancel{m}}{M+m}\right) = Px$$

or $$\frac{1}{2}mu^2\frac{M}{M+m} = Px \quad \text{---} \; (4)$$

Dividing (4) by (1) we get

$$\frac{M}{M+m} = \frac{x}{d}$$

or $$x(M+m) = Md$$

or $$x = \frac{Md}{M+m}$$ proved

PRINCIPLE OF VIRTUAL WORK 29/4/2019

If a system of forces is in equilibrium, the work done by the applied forces in a ~~virtual~~ virtual displacement compatiable with constraint is zero.

Proof

Consider a system of particles of which the i.th particle is acted upon by several forces. If the system is in equilibrium, then the sum of forces $\vec{F_i}$ acting on each particle must be zero

i.e $\vec{F_i}=0$

The work done by the force $\vec{F_i}$ in a virtual displacement $\delta\vec{r_i}$ must be zero i.e

$$\vec{F_i}\cdot\delta\vec{r_i}=0$$

Summing it over all particles of the system, we have

$$\sum_i \vec{F_i}\cdot\delta\vec{r_i}=0 \quad \text{---} (1)$$

Now we divide the force $\vec{F_i}$ into an applied force $\vec{F_i}^{(a)}$ and the forces of constrain $\vec{f_i}$. Then from (1) we have $\sum_i(\vec{F_i}^{(a)}+\vec{f_i})\cdot\delta\vec{r_i}=0$

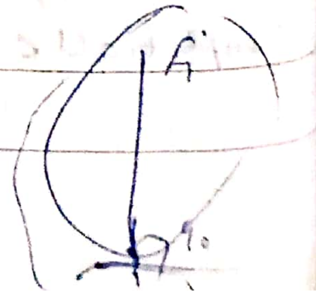
or

$$\sum_i \vec{F_i}^{(a)}\cdot\delta\vec{r_i}+\sum_i\vec{f_i}\cdot\delta\vec{r_i}=0 \quad \text{--} (2)$$

Now we consider a system in which the virtual work done by the forces of constrains is zero. for an example such a system as if we consider a particle is constrained to move on a smooth surface so that the forces of constraints are perpendicular to the surface and the displacement is tangentials to it. Then the work done by forces of constraints is zero

i.e $\sum_i \vec{f_i}\cdot\delta\vec{r_i}=0$ -- (3)

from (2) and (3) we get



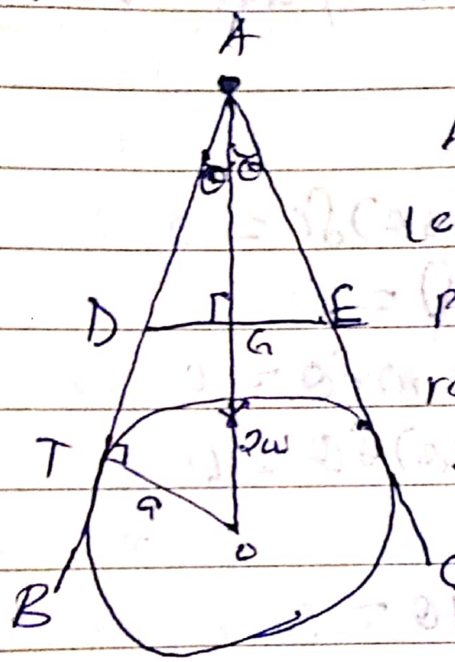
$$\sum_i \vec{F_i}^{(a)} \cdot \delta\vec{r_i} = 0$$

Hence the theorem □

problems

Q(1) Two equal uniform rods AB and AC, each of length 2b, are freely jointed at A and rest on a smooth vertical circle of radius a. If 2θ be angle between them prove that $b\sin^3\theta = a\cos\theta$

Sln



AB = AC = 2b

Let D and E be mid point of AB and AC respectively

$\therefore AD = \frac{1}{2}AB = \frac{2b}{2} = b$

$AE = \frac{1}{2}AC = b$

Let G be the centre of gravity at which the total weight 2w of two rods acts vertically downwords

in ΔADE

$\sin\theta = \frac{DG}{AD} = \frac{DG}{b}$ $\therefore DG = b\sin\theta$ i.e $DE = 2b\sin\theta$

$\cos\theta = \frac{AG}{AD} = \frac{AG}{b}$ $\therefore AG = b\cos\theta$ --- ①

In ΔATO

$\sin\theta = \frac{OT}{OA} = \frac{a}{OA}$

$\therefore OA = \frac{a}{\sin\theta} = a\operatorname{cosec}\theta \quad \text{--} \ (2)$

Let the system undergo a slight symmetrical virtual displacement about AO so that θ changes to $\theta+\delta\theta$

Then the equation of virtual work when the centre O is taken as a fixed point can be written as

$-2W\delta(OG) = 0$

$-2W\delta(OA - GA) = 0$

$-2W\delta(a\operatorname{cosec}\theta - b\cos\theta)\delta\theta = 0$

$\Rightarrow \delta(a\operatorname{cosec}\theta - b\cos\theta) = 0$

$-a\operatorname{cosec}\theta\cot\theta\,\delta\theta + b\sin\theta\,\delta\theta = 0$

$(b\sin\theta - a\operatorname{cosec}\theta\cot\theta).\delta\theta = 0$

$\because \delta\theta \neq 0$

$\therefore b\sin\theta - a\operatorname{cosec}\theta\cot\theta = 0$

or $b\sin\theta = a\operatorname{cosec}\theta\cot\theta$

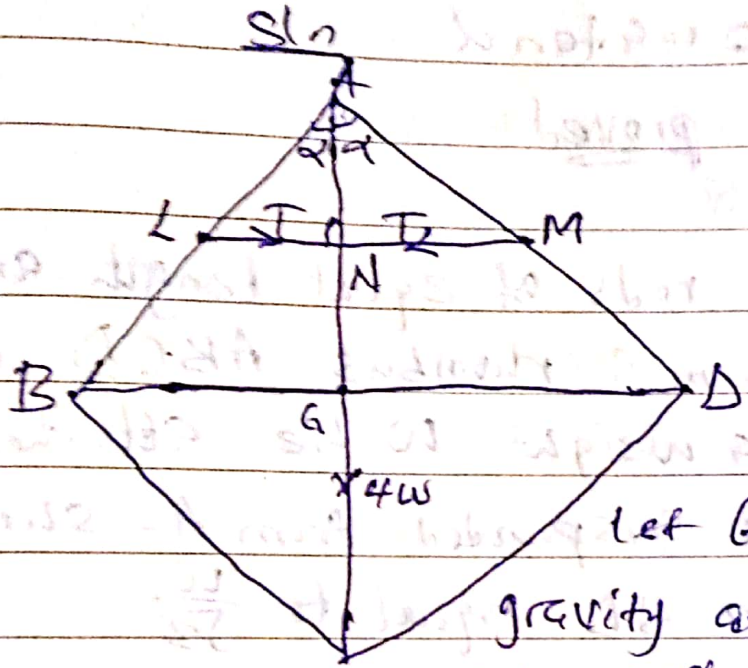
or $b\sin\theta = \frac{a}{\sin\theta} \cdot \frac{\cos\theta}{\sin\theta}$

or $b\sin^3\theta = a\cos\theta$

proved//

3/5/2019

Q(2) four equal heavy uniform rods are freely jointed so as to form a rhombus which is freely suspended by one angular point and the middle points of two upper rods are connected by a light rod so that the rhombus cannot collapse. prove that the tension of this light rod is $4w\tan\alpha$, where w is the weight of each rod and 2α is the angle of the rhombus at the point of suspension.



Let $AB = BC = CD = DA = a$

Let G be the centre of gravity at which the total weight 4W of the four rods acts.

Let T be the tension in the light rod LM where L and M are mid points of AB and AD respectively

in ΔANL, $AL = a/2$ $\quad \sin\alpha = \frac{LN}{AL} = \frac{LN}{a/2}$

$\therefore LN = \frac{a}{2}\sin\alpha$

Let the system undergo a small symmetrical displacement about the vertical AG such that α changes to $\alpha + \delta\alpha$ and the length of the rod LM alters

Taking A as the reference points, the equation of virtual work can be written as

$4w\,\delta(AG) + T\,\delta(LM) = 0$

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$\Rightarrow 4w\delta(a\cos\alpha) + T\delta(a\sin\alpha) = 0$

$\Rightarrow -4wa\sin\alpha\,\delta\alpha + Ta\cos\alpha\,\delta\alpha = 0$

$\Rightarrow a[T\cos\alpha - 4w\sin\alpha]\delta\alpha = 0$

$\because \delta\alpha \neq 0,\ a \neq 0$

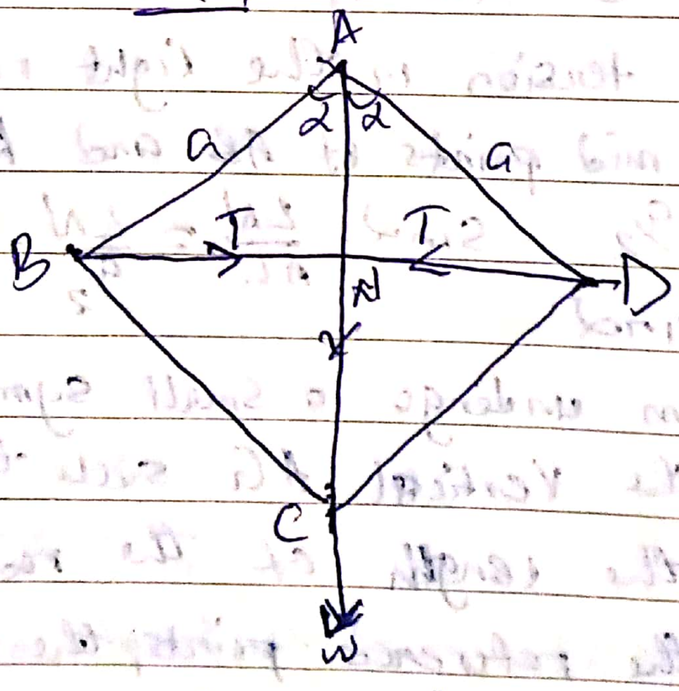
$\therefore T\cos\alpha = 4w\sin\alpha$

$T = 4w\dfrac{\sin\alpha}{\cos\alpha} = 4a\tan\alpha$

proved

Q(3) Five weightless rods of equal length are jointed together so as to form a rhombus ABCD with one diagonal BD. If a weight W be attached to C and the system be suspended from A. show that there is thrust in BD equal to $\frac{W}{\sqrt{3}}$

sln



since the forces acting at A is balanced by the weight w at C, therefore AC is vertical and BD

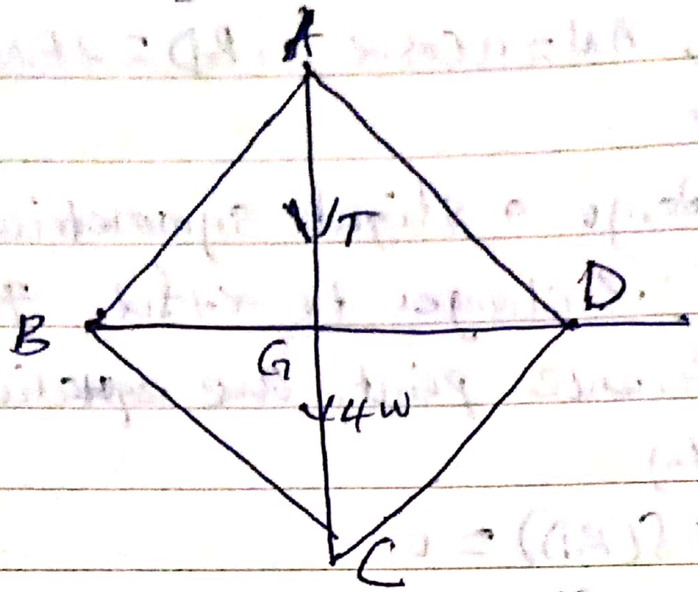
is horizontal.
Let T be the thrust in BD. Let a be the length of each rod.
Then, $BN = a\sin\alpha$, $AN = a\cos\alpha$ $\therefore BD = 2BN = 2a\sin\alpha$, $AC = 2AN = 2a\cos\alpha$
Let the system undergo a slight symmetrical displacement about AC so that α changes to $\alpha + \delta\alpha$. ~~Taking as~~
Taking A as reference point, the equation of virtual work is given by

$$W\delta(AC) + T\delta(BD) = 0$$
$$W\delta(2a\cos\alpha) + T\delta(2a\sin\alpha) = 0$$
$$\Rightarrow -W2a\sin\alpha\,\delta\alpha + T2a\cos\alpha\,\delta\alpha = 0$$
$$\Rightarrow 2a(T\cos\alpha - W\sin\alpha)\delta\alpha = 0$$
$$\because \delta\alpha \neq 0,\ 2a \neq 0$$
$$\therefore T\cos\alpha = W\sin\alpha$$
$$\therefore T = W\tan\alpha$$

In the equilibrium position ΔABD is an equilateral triangle $\therefore \alpha = 30°$
so that $T = W\tan 30°$
$$= \frac{W}{\sqrt{3}}$$
Proved

Q(4) Four uniform rods are freely jointed at their extremities and form a parallelogram ABCD which is suspended by the joint A and is kept in shape by a string AC. prove that the tension of the string is equal to half the weight of the four rods.

Sln



Let G be the centre of gravity at which the total weight 4W of the rods acts.

Let T be the tension in the string AC. Let AG=x so that AC=2x. Let the system undergo a slight symmetrical displacement about AC such that x changes to x+δx. Taking A as reference point, the eqn. of virtual work done is

~~$\because \delta x \neq 0 \therefore 4W - 2T = 0$~~

~~or $2T = 4W$~~

~~or $T = \frac{1}{2}(4W)$~~

$4W\delta(AG) - T\delta(AC) = 0$

$4W\delta(x) - T\delta(2x) = 0$

$4W\delta x - T2\delta x = 0$

$(4W - 2T)\delta x = 0$

$\because \delta x \neq 0 \therefore 4W - 2T = 0$

or $2T = 4W$

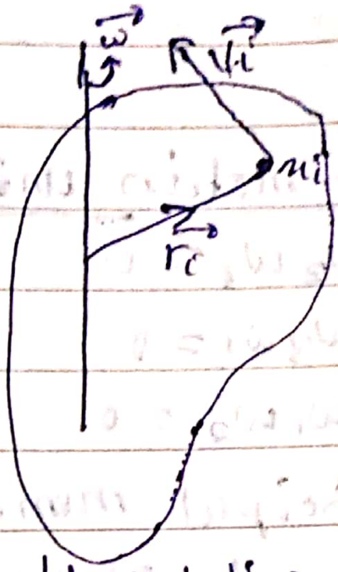
or $T = \frac{1}{2}(4W)$

= half of the weight of the fours rods

proved

Rigid Body Motion

Rotational Kinetic Energy of a rigid body body.



consider a rigid body rotating with angular velocity $\vec{\omega}$ about an axis passing through a point O fixed in the body Let $\vec{r_i}$ and $\vec{V_i}$ be the position and velocity vectors of the i-th mass m_i w.r.t the point O Then we have

$$\vec{V_i} = \vec{\omega} \times \vec{r_i} \quad \text{---} (1)$$

The rotational Kinetic energy

$$T = \sum_i \frac{1}{2} m_i V_i^2$$

$$T = \frac{1}{2} \sum_i m_i \vec{V_i} \cdot \vec{V_i}$$

$$= \frac{1}{2} \sum_i m_i \vec{V_i} \cdot (\vec{\omega} \times \vec{r_i}) \quad \text{by } (1)$$

$$= \frac{1}{2} \sum_i m_i \vec{\omega} (\vec{r_i} \times \vec{V_i}) \quad (\text{by } \vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a})$$

by the property of a scaler tripple product.

$$= \frac{1}{2} \vec{\omega} \sum_i (\vec{r_i} \times m_i \vec{V_i})$$

$\vec{a} \cdot \vec{a} = \vec{a}^2 = a^2$ $\quad \vec{a} \cdot (\vec{b} \times \vec{c})$

$= \vec{b} \cdot (\vec{c} \times \vec{a}) = c \cdot (\vec{a} \times \vec{b})$

$T = \frac{1}{2}\vec{\omega}\cdot\vec{L}$

where $\vec{L} = \sum_i \vec{r_i} \times m_i\vec{v_i}$ is the angular momentum of the rigid body

<u>Euler's Equation of motion under no external force</u>

$$I_1\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3 = 0$$

$$I_2 \omega_2 - (I_3 - I_1)\,\omega_3\omega_1 = 0$$

$$I_3 \omega_3 - (I_1 - I_2)\,\omega_1\omega_2 = 0$$

I_1, I_2, I_3 are principal moments of Inertia.

$\omega_1, \omega_2, \omega_3$ are components of angular velocity

$\vec{\omega} = \omega_1\vec{i} + \omega_2\vec{j} + \omega_3\vec{k}$

Problems 13|5|2019

Q(1) A Cube with one corner fixed is moving under no external forces. If $\omega_1, \omega_2, \omega_3$ are the angular velocities about the three edges meeting at a point, prove that $\omega_1+\omega_2+\omega_3$ and $\omega_1^2+\omega_2^2+\omega_3^2$ are both constant.

Sln

The Euler's equations of motion under no external forces are given by

$$\left.\begin{aligned} I_1\dot{\omega}_1-(I_2-I_3)\omega_2\omega_3=0 \\ I_2\dot{\omega}_2-(I_3-I_1)\omega_3\omega_1=0 \\ I_3\dot{\omega}_3-(I_1-I_2)\omega_1\omega_2=0 \end{aligned}\right\} \text{---(1)}$$

for a cube $I_1=I_2=I_3$

$\therefore$ Eqn (1) became

$I_1\dot{\omega}_1=0, \ I_2\dot{\omega}_2=0, \ I_3\dot{\omega}_3=0$

integrating

$I_1\omega_1=k_1, \ I_2\omega_2=k_2, \ I_3\omega_3=k_3$

where k_1, k_2, k_3 are constant

$\therefore \omega_1+\omega_2+\omega_3=\frac{k_1}{I_1}+\frac{k_2}{I_2}+\frac{k_3}{I_3}=\text{constant}$

also $\omega_1^2+\omega_2^2+\omega_3^2=\left(\frac{k_1}{I_1}\right)^2+\left(\frac{k_2}{I_2}\right)^2+\left(\frac{k_3}{I_3}\right)^2=\text{Constant}$

proved

Q(2) A symmetrical body with moments of Inertia $I_1=I_2$ is moving under no external forces with one point

...ixed. Show that the angular speed is constant.

Sol^n

Euler's equation of motion under no external forces are given by

$$\left.\begin{aligned} I_1\dot{w}_1-(I_2-I_3)w_2w_3=0 \\ I_2\dot{w}_2-(I_3-I_1)w_3w_1=0 \\ I_3\dot{w}_3-(I_1-I_2)w_1w_2=0 \end{aligned}\right\} \text{---} (1)$$

∵ we have given that $I_1=I_2$

∴ The third equation of (1) becomes

$$I_3\dot{w}_3=0$$

Integrating $I_3w_3=\text{constant}=K_3\text{ (say)}$

i.e. $w_3=\frac{K_3}{I_3}=\text{constant}=c\text{ (say)}$

putting $w_3=c$ in the first two equation of (1) we have $I_1\dot{w}_1-(I_2-I_3)w_2c=0$

and $I_2\dot{w}_2-(I_3-I_1)w_1c=0$

$$\left.\begin{aligned} \text{or } \dot{w}_1-\frac{(I_2-I_3)}{I_1}c\,w_2=0 \\ \dot{w}_2-\frac{(I_3-I_1)}{I_2}c\,w_1=0 \end{aligned}\right\} \text{---} (2)$$

let $\frac{(I_2-I_3)}{I_1}c=k$

Then $\frac{(I_3-I_1)}{I_2}c=-\frac{(I_1-I_3)}{I_2}c=-k$ $(\because I_1=I_2)$

Hence (2) becomes

$\dot{\omega}_1 - K\omega_2 = 0$
and $\dot{\omega}_2 + K\omega_1 = 0$ —— (3)

multiplying the first equation by ω_1 and the second equation by ω_2 and then adding them we obtain

$\omega_1 \dot{\omega}_1 + \omega_2 \dot{\omega}_2 = 0$

$\Rightarrow 2\omega_1 \dot{\omega}_1 + 2\omega_2 \dot{\omega}_2 = 0$

$\Rightarrow \frac{d}{dt}(\omega_1^2 + \omega_2^2) = 0$

integrating $\omega_1^2 + \omega_2^2 = \text{constant}$

$\therefore$ Angular speed $= \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{\text{constant} + \text{constant}}$
$= \text{constant}$ proved

Constraints, Generalized coordinates and Degrees of freedom. Kinetic Energy, Lagrange's Equation of motion, Hamilton principle

1. Constraints: If in the study of motion of a particle or a system of particles relative to an inertial system of coordinates we impose some restrictions of geometrical nature on the position and velocity of the particles, then these restrictions are called constraints. Constraints restrict the particle's motion to occure in a specialized manner.

2. Generalized Coordinates: All those independent variables which are required to specify the position of a particle or a system of particles are called generalized coordinates.

3. Degrees of freedom: The minimum number of independent

...iriables required to determine the position of a partic... or a system of particles is called the degrees of freed...
for example:

17/5/2019

Kinetic Energy of a dynamical system (in terms of generalized coordinates):

Consider a dynamical system consisting of N particles each with a constant mass m_i $(i = 1, 2, \dots N)$ and at any time t the position vector $\vec{r_i}$. Let the motions of this system be described n generalized coordinates q_j $(j = 1, 2, \dots n)$. Then we have

$$\vec{r_i} = \vec{r_i}(q_1, q_2, q_3, \dots t) = \vec{r_i}(q_j, t)$$

The velocity is given by

$$\dot{\vec{r_i}} = \frac{d\vec{r}}{dt} = \frac{\partial \vec{r_i}}{\partial q_1}\dot{q}_1 + \frac{\partial \vec{r_i}}{\partial q_2}\dot{q}_2 + \frac{\partial \vec{r_i}}{\partial q_3}\dot{q}_3 + \dots + \frac{\partial \vec{r_i}}{\partial q_m}\dot{q}_m + \frac{\partial \vec{r_i}}{\partial t}$$

$$= \sum_{j=1}^{n} \frac{\partial \vec{r_i}}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t}$$

The kinetic energy is given by

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{r_i}}^2$$

$$\text{or } T = \sum_{i=1}^{N} \frac{1}{2} m_i \left[\sum_{j=1}^{n} \frac{\partial \vec{r_i}}{\partial q_j} + \frac{\partial \vec{r_i}}{\partial t}\right]^2$$

$$= \sum_{i=1}^{N} \frac{1}{2} m_i \left[\sum_{jk} \frac{\partial \vec{r_i}}{\partial q_j} \cdot \frac{\partial \vec{r_i}}{\partial q_k} \dot{q}_j \dot{q}_k + \cancel{\dots}\right.$$

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$$2\sum_j \frac{\partial \vec{r_i}}{\partial q_j}\cdot\frac{\partial \vec{r_i}}{\partial t}\dot{q}_j + \left(\frac{\partial \vec{r_i}}{\partial t}\right)^2 \Big]$$

$$T = \sum_{j,k} a_{jk}\dot{q}_j\dot{q}_k + \sum_j b_j \dot{q}_j + C \quad \text{--- (1)}$$

where $a_{jk} = \frac{1}{2}\sum_{i=1}^{N} m_i \frac{\partial \vec{r_i}}{\partial q_j}\cdot\frac{\partial \vec{r_i}}{\partial q_k}$

$$b_j = \sum_{i=1}^{N} m_i \frac{\partial \vec{r_i}}{\partial q_j}\cdot\frac{\partial \vec{r_i}}{\partial t}$$

$$C = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\frac{\partial \vec{r_i}}{\partial t}\right)^2$$

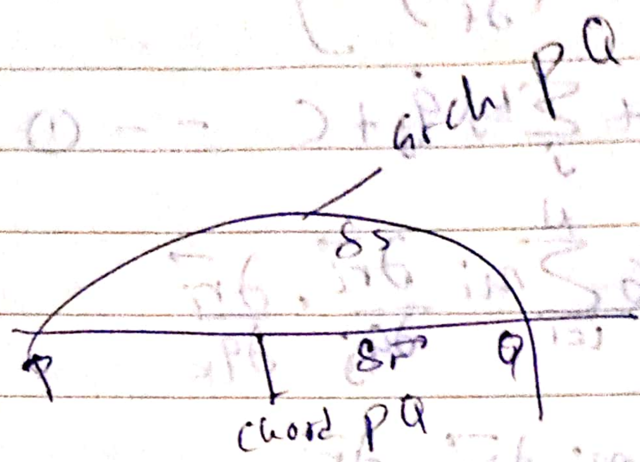
Hence (1) is the general expression for the kinetic energy of dynamical system. if the geometrical relations to not contain time t explicitly then.

$\frac{\partial \vec{r}}{\partial t} = 0$ so that $b_j = 0$ $C = 0$ and then (1) reduces to

$$T = \sum_{jk} a_{jk}\dot{q}_j\dot{q}_k$$

This is a quadratic Homogeneous expression of generalized velocities.

$+$



arch PQ $= \delta s$

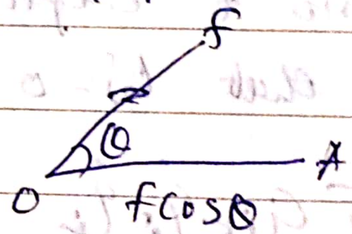
chord PQ $= \delta\vec{r}$

$\omega = \frac{d\theta}{dt} = \dot{\theta} \Rightarrow \ddot{\theta} = 0$

$F \cdot A = 2\dot{r}\dot{\theta} + r\ddot{\theta} = 2\dot{r}\omega + r \cdot 0 = 2\omega\dot{r}$

constant

$= k\dot{r}$



$W = \vec{f} \cdot \vec{d}$

$= fd\cos\theta$

work = force × displacement in the direction of force

$x^2 + y^2 =$

$x^2 + 2hxy + by^2$